

Locality of dynamics in general harmonic quantum systems

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Abstract

The Lieb-Robinson theorem states that locality is approximately preserved in the dynamics of quantum lattice systems. Whenever one has finite-dimensional constituents, observables evolving in time under a local Hamiltonian will essentially grow linearly in their support, up to exponentially suppressed corrections. In this work, we formulate Lieb-Robinson bounds for general harmonic systems on general lattices, for which the constituents are infinite-dimensional, as systems representing discrete versions of free fields or the harmonic approximation to the Bose-Hubbard model. We consider both local interactions as well as infinite-ranged interactions, showing how corrections to locality are inherited from the locality of the Hamiltonian: Local interactions result in stronger than exponentially suppressed corrections, while non-local algebraic interactions result in algebraic suppression. We derive bounds for canonical operators, Weyl operators and outline generalization to arbitrary operators. As an example, we discuss the Klein-Gordon field, and see how the approximate locality in the lattice model becomes the exact causality in the field limit. We discuss the applicability of these results to quenched lattice systems far from equilibrium, and the dynamics of quantum phase transitions.

1 Introduction

Locality in relativistic theories ensures that space-like separated observables commute: One simply cannot communicate faster than light. In non-relativistic lattice models, in contrast, there is no a priori reason for the support of time-evolved operators to stay confined within a light cone. It is one of the classic results of mathematical physics, dating back to Lieb and Robinson [1], that even in a non-relativistic quantum spin model on a lattice, locality is preserved in an approximate sense: There is always a well-defined velocity of information propagation and hence a causal, “sound” or “light” cone. Locality is then respected under quantum spin dynamics with finite-ranged interactions, in that the support of any local observable evolved for some time will remain local to a region of size linear in this time, up to a correction that is at least exponentially suppressed. Except from exponentially decaying tails, hence, one encounters a situation very much like in the relativistic setting. The Lieb-Robinson theorem has hence put the observation in many quantum lattice models on a rigorous footing that there exists a well-defined finite speed of propagation, often referred to as group velocity.

Over the years, Lieb-Robinson bounds have been extended and generalized to higher-dimensional spin systems on lattices, and the bounds have been significantly improved in several ways [1, 2, 3, 4, 5, 6, 7, 8]. Also, important new applications of the Lieb-Robinson theorem have been found: Notably, the only known proof for the clustering of correlations in gapped lattice models is based on the Lieb-Robinson theorem [2, 4], hence rigorously confirming the “folk theorem” in condensed matter physics that correlation functions decay exponentially in gapped models. “Area laws” for the scaling of entropy in ground states of one-dimensional spin systems can be proven based on this result [9, 10, 11]. Finally, in the context of quantum information theory, it provides a bound to the velocity one can transmit quantum information through a chain of systems giving rise to a quantum channel [11], a topic that has received quite some attention in the quantum information literature.

In the simplest form of the Lieb-Robinson theorem, one considers a spin system $\mathcal{H} = \bigoplus_{i \in L} \mathcal{H}_i$, $\mathcal{H}_i = \mathbb{C}^d$, on a lattice with vertices L , and a local (finite-ranged) Hamiltonian $\hat{H} = \sum_{X \subset L} \hat{\Phi}(X)$. The time evolution of an observable \hat{A} on some subset $A \subset L$ of the lattice under this Hamiltonian,¹

$$\hat{A}(t) = e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t}, \quad (2)$$

then forms a group of automorphisms. At time zero, obviously any observable \hat{A} being supported on A will commute with any observable \hat{B} that is supported on a disjoint set $B \subset L$. The Lieb-Robinson bound now gives a bound to this commutator if \hat{A} is evolved in time under this local Hamiltonian. It says that there exists a constant $C > 0$ and a “speed of light” $v > 0$ such that

$$||[\hat{A}(t), \hat{B}]|| \leq C ||\hat{A}|| ||\hat{B}|| e^{-\mu(dist(A,B) - v|t|)}, \quad (3)$$

where $dist(A, B) = \min_{a \in A, b \in B} dist(a, b)$ is the minimal distance between the two regions and $|| \cdot ||$ denotes the operator norm. In other words, outside the causal cone $v|t| < dist(A, B)$, one encounters merely an exponential tail, and the supports of $\hat{A}(t)$ and \hat{B} stay almost disjoint. Eq. (3) governs the maximum speed at which a local excitation can travel through the lattice and the maximum speed at which correlations can build up over time.

The physical setting considered here can equally be viewed as the study of the situation of quickly *quenching* from, say, a system that is in the ground state of some local Hamiltonian to a new local Hamiltonian [15, 16, 17, 18, 19, 20, 21, 22]. This setting has also been linked to the entanglement generation and scaling in quenched systems [10, 11, 12, 13, 23]. Studies of non-equilibrium dynamics of quantum lattice systems of this type are entering a renaissance recently, not the least due to experimental studies becoming more and more available. With atoms in optical lattices, for example, one can suddenly alter the system parameters and thus observe the *dynamics*

¹This automorphism group of time evolution is in the context of Lieb Robinson bounds typically denoted as

$$\tau_t^{\hat{H}}(\hat{A}) = e^{i\hat{H}t} \hat{A} e^{-i\hat{H}t}. \quad (1)$$

Since in this work, the Hamiltonian \hat{H} is always well-specified at the beginning of each section, we make use of the above notation in the Heisenberg picture for simplicity of notation.

of a quantum phase transition [24]. Hence, it seems only natural to apply the machinery of Lieb-Robinson bounds to such settings. However, despite the generality of the above mentioned results on Lieb-Robinson bounds, they are—with the exception of the recent Refs. [6]—constrained in the sense that they only apply to spin systems, so finite-dimensional constituents, quite unlike the situation encountered in many settings of non-equilibrium dynamics.

In this work, we derive Lieb-Robinson bounds to harmonic lattice systems on general graphs. Such models correspond to discrete versions of free fields, lattice vibrations, or the superfluid limit of the Bose-Hubbard model. As harmonic models, applicable merely to a class of systems, the resulting bounds can indeed be made very tight, e.g., for local Hamiltonians, we find stronger than exponential suppression, while for algebraically decaying interactions the corrections to locality are also algebraically suppressed. Within the considered class of models the very tight connection between the approximate locality of operators and the locality of the Hamiltonian is thus revealed.

These systems serve as instructive theoretical laboratories for more elaborate interacting theories for infinite-dimensional quantum systems (and Lieb-Robinson bounds have fundamental implications, e.g., in the context of using harmonic systems as quantum channels [12, 13, 14]), about which little is known when it comes to Lieb-Robinson bounds (see, however, Refs. [6, 7] for recent progress on bounded anharmonicities). In this way, we continue the program of Refs. [25, 26], building upon earlier work primarily on clustering of correlations and “*area theorems*” in harmonic lattice systems [27, 28, 29, 30, 31, 32].

This chapter is organized as follows. We first define the models under consideration, and explain what we mean by having a general lattice. We then present bounds on the time evolved canonical coordinates and make the causal cone explicit. For the important class of Weyl operators on the lattice, we explicitly find bounds on their operator norms and discuss generalizations to arbitrary operators. As an example, we discuss the case of the discrete version of the Klein-Gordon field, and show how the approximate locality in the lattice model becomes the exact locality in the continuum limit. The proofs of the main results are then presented in a separate section in great detail. As such, the findings in this work complement the findings of Ref. [6], which considers in its harmonic part nearest-neighbor interactions of translationally invariant models on cubic lattices. We finally discuss the implications to quenched many-body systems far from equilibrium.

2 Considered models and main results

We consider harmonic systems on general lattices. Such lattices are described by an undirected graph $G = (L, E)$, with vertices L and edge set E . The vertices L correspond to the physical degrees of freedom, here bosonic modes with Hilbert space $\mathcal{H}_i = \mathcal{L}^2(\mathbb{R})$, $i \in L$. Edges reflect a neighborhood relation on the lattice. On L we use the graph-theoretical distance $\text{dist}(i, j)$ between $i, j \in L$, i.e., the shortest path connecting i and j . On such a general set of lattice sites L we consider harmonic

Hamiltonians of the form

$$\hat{H} = \frac{1}{2} \sum_{i,j \in L} (\hat{x}_i X_{i,j} \hat{x}_j + \hat{p}_i P_{i,j} \hat{p}_j), \quad (4)$$

where $X_{i,j} = X_{j,i} \in \mathbb{R}$, $P_{i,j} = P_{j,i} \in \mathbb{R}$ and the \hat{x}_i, \hat{p}_i are canonical coordinates obeying the usual commutation relations (we set $\hbar = 1$) $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$, $[\hat{x}_i, \hat{p}_j] = i\delta_{i,j}$. Identifying

$$A_{i,j} = \frac{X_{i,j} + P_{i,j}}{2}, \quad B_{i,j} = \frac{X_{i,j} - P_{i,j}}{2}, \quad \hat{b}_i = \frac{\hat{x}_i + i\hat{p}_i}{\sqrt{2}}, \quad (5)$$

the above Hamiltonian is equivalent to a Hamiltonian quadratic in the annihilation and creation operators of the bosonic modes ($[\hat{b}_i, \hat{b}_j] = 0$, $[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{i,j}$) of the form

$$\hat{H} = \frac{1}{2} \sum_{i,j \in L} (\hat{b}_i^\dagger A_{i,j} \hat{b}_j + \hat{b}_i A_{i,j} \hat{b}_j^\dagger + \hat{b}_i B_{i,j} \hat{b}_j + \hat{b}_i^\dagger B_{i,j} \hat{b}_j^\dagger). \quad (6)$$

All the following results may thus also be obtained for the above Hamiltonian and bosonic operators by using the identification in Eq. (5). We suppose that L is countable such that we may identify the couplings $(C_{i,j})_{i,j \in L}$, $C = X, P$, with matrices, thereby defining the operator norm $\|C\|$ and multiplications in the usual matrix sense. We denote the time evolution of operators \hat{o} in the Heisenberg picture as

$$\hat{o}(t) = e^{i\hat{H}t} \hat{o} e^{-i\hat{H}t}, \quad (7)$$

and by $\|\cdot\|$ the operator norm throughout.

2.1 Local couplings

In this subsection, we will derive Lieb-Robinson bounds for harmonic systems with arbitrary local interactions on the graph. We will see that outside a causal cone we obtain a stronger than exponentially suppressed influence of time evolved canonical coordinates in the Heisenberg picture. In the above notation, local means that

$$X_{i,j} = P_{i,j} = 0 \text{ for } \text{dist}(i,j) > R. \quad (8)$$

For notational clarity we write

$$d_{i,j} := \text{dist}(i,j)/R, \quad \tau := \max\{\sqrt{\|PX\|}, \sqrt{\|XP\|}\}|t| \quad (9)$$

in the following, and denote by $\lceil x \rceil = \min\{z \in \mathbb{Z} \mid x \leq z\}$ the ceiling function. For a Hamiltonian as in Eq. (4) with local couplings as above we prove the following theorems.

Theorem 1 (Lieb-Robinson bounds for local couplings) *Writing $b_{i,j} = \lceil d_{i,j}/2 \rceil$ and $a_{i,j} = \max\{0, \lceil (d_{i,j} - 1)/2 \rceil\}$, one has*

$$\begin{aligned} \frac{\sqrt{\|PX\|}}{\|P\|} \|\hat{x}_i(t), \hat{x}_j\|, \frac{\sqrt{\|XP\|}}{\|X\|} \|\hat{p}_i(t), \hat{p}_j\| &\leq \frac{\tau^{2a_{i,j}+1} \cosh(\tau)}{(2a_{i,j}+1)!}, \\ \|\hat{x}_i(t), \hat{p}_j\|, \|\hat{p}_i(t), \hat{x}_j\| &\leq \frac{\tau^{2b_{i,j}} \cosh(\tau)}{(2b_{i,j})!}. \end{aligned} \quad (10)$$

We note that ($d \in \mathbb{N}$ and we use $1/d! \leq (e/d)^d d^{-1/2}$)

$$\frac{\tau^d \cosh(\tau)}{d!} \leq \frac{e^{\tau+d(1+\log(\tau)-\log(d))}}{\sqrt{d}}, \quad (11)$$

i.e., for sufficiently large $\text{dist}(i, j)$, one finds a faster-than-exponential decay. In the subsequent formulation we make this more explicit by defining a “light cone”, $e\tau < d_{i,j}$ (i.e., $c|t| < \text{dist}(i, j)$), with a “speed of light” given by

$$c := eR \max\{\|XP\|^{1/2}, \|PX\|^{1/2}\}. \quad (12)$$

Then commutators of “space-like separated” operators are strongly suppressed. This c is an upper bound to the speed with which a local excitation would travel through the lattice in a non-equilibrium situation. This kind of argument is used, e.g., in Ref. [15], where a central limit type argument was used to show exact relaxation in a quenched system – the intuition being that inside the cone excitations randomize the system while the influence of excitations outside the cone is negligible, which is essentially a Lieb-Robinson-type argument. In a very similar fashion, one can argue that the speed at which correlations build up in time is governed by these bounds (see also Ref. [11]). Again, the above speed is an upper bound to which one can correlate separate regions starting from an uncorrelated state under quenched, non-equilibrium dynamics.

As mentioned before, these bounds have immediate implication to the evaluation of capacities of harmonic chains, when being used as convenient quantum channels. Such an idea of *transporting quantum information* through interacting quantum systems is an appealing one, as no exact local control is required, and the transport of quantum information is merely due to the free evolution of an excitation under local dynamics.² Following the argument of Ref. [11], one could use such a harmonic chain as a quantum channel through which one sends classical information, encoded in the application of a local unitary at some site, and letting the system freely evolve in time. The decoding corresponds to a readout at a distant site. Then indeed, outside the cone defined by the “speed of light”, the *classical information capacity* C of this quantum channel is exponentially small. This means that the classical information capacity of harmonic chains used as quantum channels is – for a fixed time – exponentially small in the distance between sender and observer. The causal cone is made even more explicit in the subsequent formulation.

Theorem 2 (Alternate version making the causal cone explicit) *Let $e\tau < d_{i,j}$. Then*

$$\frac{\sqrt{\|PX\|}}{\|P\|} \|\hat{x}_i(t), \hat{x}_j\|, \frac{\sqrt{\|XP\|}}{\|X\|} \|\hat{p}_i(t), \hat{p}_j\|, \|\hat{x}_i(t), \hat{p}_j\|, \text{ and } \|\hat{p}_i(t), \hat{x}_j\| \quad (13)$$

are all bounded from above by

$$\frac{e^{d_{i,j} \log(e\tau/d_{i,j})}}{\sqrt{d_{i,j}} \left(1 - (e\tau/d_{i,j})^2\right)}. \quad (14)$$

²For harmonic instances, see, e.g., Refs. [12, 13, 14], but there is a vast literature also for spin systems and other finite-dimensional quantum systems, to mention all of which would be beyond the scope of this chapter.

Often, X and P will commute, rendering the max in Eqs. (9,12) irrelevant. In many physical situations one even has $P_{i,j} = \delta_{i,j}$, i.e., $\tau = \sqrt{\|X\|}|t|$, in which case the above bounds may be improved, in particular, the “speed of light” improves to $c = eR\|X\|^{1/2}/2$. For clarity, we explicitly state the new bounds in this case, in both ways.

Theorem 3 (Lieb-Robinson bounds for local couplings and $P = 1$) *Let $a_{i,j} = \max\{0, \lceil d_{i,j} - 1 \rceil\}$. Then*

$$\begin{aligned} \sqrt{\|X\|} \|\hat{x}_i(t), \hat{x}_j\| &\leq \frac{\tau^{2\lceil d_{i,j} \rceil + 1} \cosh(\tau)}{(2\lceil d_{i,j} \rceil + 1)!}, \\ \frac{1}{\sqrt{\|X\|}} \|\hat{p}_i(t), \hat{p}_j\| &\leq \frac{\tau^{2a_{i,j} + 1} \cosh(\tau)}{(2a_{i,j} + 1)!}, \\ \|\hat{x}_i(t), \hat{p}_j\|, \|\hat{p}_i(t), \hat{x}_j\| &\leq \frac{\tau^{2\lceil d_{i,j} \rceil} \cosh(\tau)}{(2\lceil d_{i,j} \rceil)!}. \end{aligned} \quad (15)$$

Theorem 4 (Alternate version making the causal cone explicit for $P = 1$) *For $e\tau < 2d_{i,j}$ one has*

$$\sqrt{\|X\|} \|\hat{x}_i(t), \hat{x}_j\|, \|\hat{x}_i(t), \hat{p}_j\|, \|\hat{p}_i(t), \hat{x}_j\| \leq \frac{e^{2d_{i,j} \log(e\tau/(2d_{i,j}))}}{\sqrt{d_{i,j}} \left(1 - (e\tau/(2d_{i,j}))^2\right)}, \quad (16)$$

and for $e\tau < (2a_{i,j} + 1)$

$$\frac{1}{\sqrt{\|X\|}} \|\hat{p}_i(t), \hat{p}_j\| \leq \frac{e^{2a_{i,j} \log(e\tau/(2a_{i,j} + 1))}}{\sqrt{a_{i,j}} \left(1 - (e\tau/(2a_{i,j} + 1))^2\right)}, \quad (17)$$

where now $a_{i,j} = \max\{0, \lceil d_{i,j} - 1 \rceil\}$.

2.2 Application: Non-relativistic quantum mechanics yields causality in the field limit

This section forms an application of the previous considerations. We will see how the exact light cone of the free field is recovered from the approximate light cone in the Lieb-Robinson theorem in the continuum limit of the lattice version of the field theory. It is very instructive indeed to see how the tails in the superexponentially suppressed region outside the light cone becomes more and more suppressed in this limit. The role of the Lieb-Robinson velocity is hence taken over by the speed of light in the relativistic model.

We start from the *Klein-Gordon Hamiltonian* on $V = [0, 1]^{\times D}$ in units $\hbar = c = 1$,

$$\hat{H} = \frac{1}{2} \int_V d\mathbf{x} \left[\hat{\pi}^2(\mathbf{x}) + \sum_{d=1}^D (\partial_{x_d} \hat{\varphi}(\mathbf{x}))^2 + m^2 \hat{\varphi}^2(\mathbf{x}) \right], \quad (18)$$

where the field operators fulfill the usual commutation relations

$$[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \quad [\hat{\varphi}(\mathbf{x}), \hat{\varphi}(\mathbf{y})] = [\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})] = 0. \quad (19)$$

Discretizing according to $\mathbf{x} = \mathbf{i}/N$, $\mathbf{i} \in \{\mathbf{j} \in \mathbb{N}^D \mid i_d = 1, 2, \dots, N\} =: L$,

$$\int_V d\mathbf{x} \hat{f}(\mathbf{x}) \rightarrow \frac{1}{N^D} \sum_{\mathbf{i} \in L} \hat{f}(\mathbf{i}/N), \quad (\partial_{x_d} \hat{f})(\mathbf{x}) \rightarrow \frac{\hat{f}(\mathbf{x} + \mathbf{n}_d/N) - \hat{f}(\mathbf{x})}{1/N}, \quad (20)$$

where \mathbf{n}_d denotes a unit vector in direction d , we find (equipping L with periodic boundary conditions),

$$\begin{aligned} \hat{H} &\rightarrow \frac{1}{2N^D} \sum_{\mathbf{i} \in L} \left[\hat{\pi}^2(\mathbf{i}/N) + \sum_{d=1}^D \left(\frac{\hat{\varphi}(\mathbf{i}/N + \mathbf{n}_d/N) - \hat{\varphi}(\mathbf{i}/N)}{1/N} \right)^2 + m^2 \hat{\varphi}^2(\mathbf{i}/N) \right] \\ &= \frac{1}{2N^D} \left(\sum_{\mathbf{i} \in L} \left[\hat{\pi}^2(\mathbf{i}/N) + (m^2 + 2DN^2) \hat{\varphi}^2(\mathbf{i}/N) \right] - N^2 \sum_{\substack{\mathbf{i}, \mathbf{j} \in L \\ \text{dist}(\mathbf{i}, \mathbf{j})=1}} \hat{\varphi}(\mathbf{i}/N) \hat{\varphi}(\mathbf{j}/N) \right) \\ &=: \hat{H}_N. \end{aligned} \quad (21)$$

Then $N \rightarrow \infty$ is the valid continuum limit for a fixed $V = [0, 1]^{\times D}$. Now,

$$\hat{x}_{\mathbf{i}} := N^{-D/2} \hat{\varphi}(\mathbf{i}/N), \quad \hat{p}_{\mathbf{i}} := N^{-D/2} \hat{\pi}(\mathbf{i}/N), \quad (22)$$

define harmonic position and momentum operators satisfying the canonical commutation relations, in terms of which we find

$$\hat{H}_N = \frac{1}{2} \sum_{\mathbf{i}, \mathbf{j} \in L} \left[\hat{p}_{\mathbf{i}} P_{\mathbf{i}, \mathbf{j}} \hat{p}_{\mathbf{j}} + \hat{x}_{\mathbf{i}} X_{\mathbf{i}, \mathbf{j}} \hat{x}_{\mathbf{j}} \right], \quad (23)$$

where $P_{\mathbf{i}, \mathbf{j}} = \delta_{\mathbf{i}, \mathbf{j}}$ and

$$X_{\mathbf{i}, \mathbf{j}} = (m^2 + 2DN^2) \delta_{\mathbf{i}, \mathbf{j}} - N^2 \delta_{\text{dist}(\mathbf{i}, \mathbf{j}), 1}. \quad (24)$$

We are interested in the discretized version of the commutator $[\hat{\varphi}(\mathbf{x}, t), \hat{\varphi}(\mathbf{0}, 0)]$, which is given by

$$N^D [\hat{x}_{\mathbf{i}}(t), \hat{x}_{\mathbf{0}}], \quad (25)$$

and set out to apply Theorem 4. We have $R = 1$ and assume w.l.o.g. that $0 \leq i_d \leq N/2$, i.e., $d_{\mathbf{i}, \mathbf{0}} = \text{dist}(\mathbf{i}, \mathbf{0}) = \sum_{d=1}^D i_d = N \sum_{d=1}^D x_d \geq N|\mathbf{x}|$, with $|\mathbf{x}|$ being the euclidean norm. Furthermore, as we assumed translational invariance the eigenvalues $\lambda_{\mathbf{k}}$ of X are given by

$$\lambda_{\mathbf{k}} = m^2 + 2DN^2 - 2N^2 \sum_{d=1}^D \cos(2\pi k_d/N), \quad (26)$$

i.e., $\|X\| \leq m^2 + 4DN^2$ (for even N we have equality).

Now fix $|t|$ and $|\mathbf{x}|$ such that

$$e\sqrt{D}|t| < |\mathbf{x}|. \quad (27)$$

We then take the limit $N \rightarrow \infty$ such that $\sum_{d=1}^D x_d = d_{\mathbf{i},0}/N = \text{const.}$ is fulfilled for all N (e.g., $\mathbf{x} = (1/4, 0, \dots, 0)$ fixes $\mathbf{i} = (N/4, 0, \dots, 0)$ and $N/4 \in \mathbb{N}$). Now let $N_0 \in \mathbb{N}$ be such that

$$1 > \frac{e|t|\sqrt{D}}{|\mathbf{x}|} \sqrt{\frac{m^2}{4DN_0^2} + 1} =: z, \quad (28)$$

which yields

$$e\tau < |\mathbf{x}| \frac{\sqrt{\|X\|}}{\sqrt{\frac{m^2}{4N_0^2} + D}} \leq 2d_{\mathbf{i},0} \frac{\sqrt{\|X\|}}{N \sqrt{\frac{m^2}{N_0^2} + 4D}} < 2d_{\mathbf{i},0} \quad (29)$$

for all $N > N_0$. This enables us to apply Theorem 4 to find

$$\|[\hat{x}_{\mathbf{i}}(t), \hat{x}_{\mathbf{0}}]\| \leq \frac{e^{2N|\mathbf{x}| \log(z)}}{\sqrt{N\|X\||\mathbf{x}|}(1-z^2)} \quad (30)$$

for all $N > N_0$, i.e.,

$$\lim_{N \rightarrow \infty} N^D \|[\hat{x}_{\mathbf{i}}(t), \hat{x}_{\mathbf{0}}]\| = 0 \quad (31)$$

independent of m . Eq. (31) shows that the approximate light cone of the Lieb-Robinson bound becomes an exact light cone in the continuum limit. The exponentially suppressed tails vanish, and approximate locality is replaced by an exact locality. It is interesting to see how this concept emerges from the bounds to the speed of information propagation in the sense of Lieb-Robinson.

The bound in Theorem 4 is not quite strong enough to recover the exact prefactor of the light cone $|t| < |\mathbf{x}|$. This is mainly due to the fact that we allowed for general lattices in the Lieb-Robinson bound. Demanding translational invariance would allow for slightly stronger bounds. In Fig. 1 we depict exact numerical results for this geometrical setting in $D = 1$.

2.3 Non-local couplings

The previous section allowed for arbitrary local interactions. In this section we will turn to strongly decaying *non-local couplings* of the form

$$|X_{i,j}|, |P_{i,j}| \leq \frac{c_0}{(\text{dist}(i,j) + 1)^\eta}. \quad (32)$$

We define the spatial dimension of L in the usual sense: For all spheres $S_r(i) \subset L$ with radius $r \in \mathbb{N}$ centered at $i \in L$,

$$S_r(i) = \{k \in L | \text{dist}(k, i) = r\}, \quad (33)$$

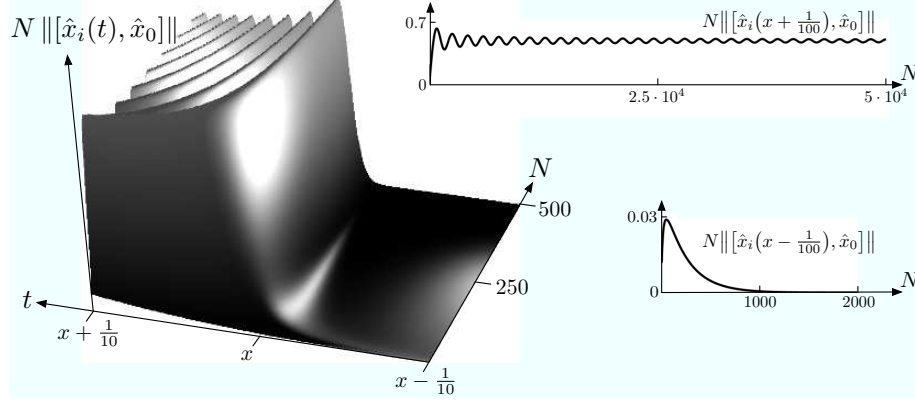


Figure 1: The light cone in the field limit of the discrete Klein-Gordon field: Depicted is $N \|\hat{x}_i(t), \hat{x}_0\|$ as a function of t and N . This is the discrete version of $\|\hat{\varphi}(x, t), \hat{\varphi}(0, 0)\|$, $x = i/N$, in one spatial dimension. For $|t| < |x|$ we find for finite system size N the Lieb-Robinson exponential decay in $|x|$ and for $N \rightarrow \infty$ the commutator $N \|\hat{x}_i(t), \hat{x}_0\|$ goes to zero for space-like separations, thus recovering exact causality.

there exists a smallest $D > 0$ such that for all $0 < r \in \mathbb{N}$

$$\sup_{i \in L} |S_r(i)| \leq c_D r^{D-1} \quad (34)$$

for some constant $c_D > 0$. This number D is taken as the dimension of the lattice. We find that the decay of interactions is inherited by the decay of the operator norm of the commutator of canonical coordinates. The same power in the exponent as in the interaction again appears in the Lieb-Robinson bounds. Note the (not accidental) similarity with the inheritance of the decay of correlation functions dependent on the decay of interactions in Ref. [25].

Theorem 5 (Bounds for non-local couplings) *Let $\eta > D$. Then*

$$\begin{aligned} \|\hat{x}_i(t), \hat{x}_j\|, \|\hat{p}_i(t), \hat{p}_j\| &\leq \frac{\sinh(\tau)}{a_0(1 + \text{dist}(i, j))^\eta}, \\ \|\hat{x}_i(t), \hat{p}_j\|, \|\hat{p}_i(t), \hat{x}_j\| &\leq \delta_{i,j} + \frac{\cosh(\tau)}{a_0(1 + \text{dist}(i, j))^\eta}, \end{aligned} \quad (35)$$

where $\tau = a_0 c_0 |t|$, $a_0 = c_D 2^{\eta+1} \zeta(1 - D + \eta)$, and ζ is the Riemann zeta function.

2.4 Weyl operators

A class of operators that play a central role in harmonic systems are the Weyl operators. Denoting the support of a Weyl operator \hat{W}_ξ as $\Xi \subset L$ it may be written as

$$\hat{W}_\xi = e^{i \sum_{i \in \Xi} (p_i \hat{x}_i - x_i \hat{p}_i)}, \text{ where } \xi = (x_1, \dots, x_{|\Xi|}, p_1, \dots, p_{|\Xi|}) \in \mathbb{R}^{2|\Xi|}. \quad (36)$$

Via the Fourier-Weyl relation general bounded operators may be expressed in terms of these operators, see below.

We define the distance of two sets $A, B \subset L$ as

$$\text{dist}(A, B) = \min_{i \in A, j \in B} \text{dist}(i, j), \quad (37)$$

and the surface area of a set $A \subset L$ as $|\partial A|$, where

$$\partial A = \{i \in A \mid \exists j \in L \setminus A : \text{dist}(i, j) = 1\} \quad (38)$$

defines the set of lattice sites on the surface of A . The following theorem establishes a connection between commutators of Weyl operators and previously derived bounds on the canonical coordinates. Note in the subsequent theorem the dependence on the right hand side of the operator norms $\|\xi\|, \|\xi'\|$ of ξ and ξ' , whereas on the left hand side of Eq. (40), we have the operator norm for commutators of Weyl operators.

Theorem 6 (Lieb-Robinson bounds for Weyl operators) *Let*

$$\hat{W}_\xi = e^{i \sum_{i \in \Xi} (p_i \hat{x}_i - x_i \hat{p}_i)}, \quad \hat{W}_{\xi'} = e^{i \sum_{i \in \Xi'} (p_i \hat{x}_i - x_i \hat{p}_i)} \quad (39)$$

be Weyl operators as defined above. Then

$$\begin{aligned} \left\| [\hat{W}_\xi(t), \hat{W}_{\xi'}] \right\| &\leq \|\xi\| \|\xi'\| \sum_{i \in \Xi, j \in \Xi'} (\| [x_i(t), x_j] \| + \| [x_i(t), p_j] \| \\ &\quad + \| [p_i(t), x_j] \| + \| [p_i(t), p_j] \|) \\ &\leq c_D \|\xi\| \|\xi'\| \min \{ |\partial \Xi|, |\partial \Xi'| \} \\ &\quad \times \sum_{d=\text{dist}(\Xi, \Xi')}^{\infty} f(d) d^{D-1} (1 + c_D(d - \text{dist}(\Xi, \Xi'))^D). \end{aligned} \quad (40)$$

where $f : \mathbb{N} \rightarrow \mathbb{R}$ is a function such that

$$\| [x_i(t), x_j] \| + \| [x_i(t), p_j] \| + \| [p_i(t), x_j] \| + \| [p_i(t), p_j] \| \leq f(\text{dist}(i, j)). \quad (41)$$

Employing, e.g., Theorem 2, we have for $e\tau < \text{dist}(\Xi, \Xi')/R =: d_{\Xi, \Xi'}$ that

$$\left\| [\hat{W}_\xi(t), \hat{W}_{\xi'}] \right\| \leq C \min \{ |\partial \Xi|, |\partial \Xi'| \} g \left(\frac{e\tau}{d_{\Xi, \Xi'}} \right) e^{d_{\Xi, \Xi'} \log(e\tau/d_{\Xi, \Xi'})} d_{\Xi, \Xi'}^{D-3/2}, \quad (42)$$

where

$$C = R^{D-1} c_D \|\xi\| \|\xi'\| \left(\frac{\|P\|}{\sqrt{\|PX\|}} + \frac{\|X\|}{\sqrt{\|XP\|}} + 2 \right) \quad (43)$$

and the function $g : (0, 1) \rightarrow \mathbb{R}$,

$$g(z) = \frac{1}{1-z^2} \sum_{d=0}^{\infty} z^{d/R} (d+1)^{D-1} (1 + c_D(d+1)^D) \geq 0, \quad (44)$$

is increasing in z with $\lim_{z \rightarrow 0} = 1$.

Note that we have in Eq. (41) expressed this statement in terms of a function f that grasps the decay of operator norms of commutators of canonical coordinates. Whenever one can identify such a function, e.g., through theorems 1-5, a result on Weyl operators can be deduced. Needless to say, in the same way we have applied Theorem 2, we could have made use of Theorem 5: Essentially, the decay in operator norms of canonical coordinates is inherited by the expression for Weyl operators. Due to the sum in Eq. (40), however, it can happen that no decay follows for Weyl operators if the (i) dimension of the lattice is too high or (ii) the decay of f is too slow. For finite-dimensional lattices, however, sufficiently fast algebraically decaying (i.e., sufficiently large η) interactions yield an algebraic Lieb-Robinson-type statement for the commutator of two Weyl operators.

Note also that only the surface areas of the two sets Ξ, Ξ' enter the bound, but not the cardinality of the support. This allows for infinite regions (for $D = 1$ both may be supported on infinite intervals, for $D > 1$ only one of Ξ, Ξ' needs to have a finite surface area) separated by $\text{dist}(\Xi, \Xi')$.

2.5 More general operators

A general bounded operator \hat{o} supported on $\Xi \subset L$ may be expressed as

$$\hat{o} = \frac{1}{(2\pi)^{|\Xi|}} \int_{\mathbb{R}^{2|\Xi|}} d\xi \chi_{\hat{o}}(-\xi) \hat{W}_{\xi}, \quad (45)$$

where

$$\chi_{\hat{o}}(\xi) = \text{tr} [\hat{o} \hat{W}_{\xi}] \quad (46)$$

is the characteristic function of \hat{o} . This allows to deduce bounds for general bounded operators using the bounds on Weyl operators stated above:

$$\|[\hat{o}(t), \hat{o}']\| \leq \frac{1}{(2\pi)^{|\Xi|+|\Xi'|}} \int_{\mathbb{R}^{2|\Xi|}} d\xi \int_{\mathbb{R}^{2|\Xi'|}} d\xi' |\chi_{\hat{o}}(-\xi) \chi_{\hat{o}'}(-\xi')| \left\| [\hat{W}_{\xi}(t), \hat{W}_{\xi'}] \right\|. \quad (47)$$

Bounds for more general, possibly unbounded, operators that are finite sums of finite products of canonical operators (or bosonic creation and annihilation operators) may be obtained by repeatedly employing operator identities such as

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A} [\hat{B}, \hat{C}] + [\hat{A}, \hat{C}] \hat{B}, \quad [\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}] \hat{C} + \hat{B} [\hat{A}, \hat{C}], \quad (48)$$

e.g., bosonic density-density commutators may be written as

$$\begin{aligned} [\hat{n}_i(t), \hat{n}_j] &= \hat{b}_i^\dagger(t) [\hat{b}_i(t), \hat{n}_j] + [\hat{b}_i^\dagger(t), \hat{n}_j] \hat{b}_i(t) \\ &= \hat{b}_i^\dagger(t) \left([\hat{b}_i(t), \hat{b}_j^\dagger] \hat{b}_j + \hat{b}_j^\dagger [\hat{b}_i(t), \hat{b}_j] \right) \\ &\quad + \left([\hat{b}_i^\dagger(t), \hat{b}_j^\dagger] \hat{b}_j + \hat{b}_j^\dagger [\hat{b}_i^\dagger(t), \hat{b}_j] \right) \hat{b}_i(t), \end{aligned} \quad (49)$$

which yields

$$\begin{aligned} |\langle [\hat{n}_i(t), \hat{n}_j] \rangle| &\leq \left| \langle \hat{b}_i^\dagger(t) \hat{b}_j \rangle \right| \left\| [\hat{b}_i(t), \hat{b}_j^\dagger] \right\| + \left| \langle \hat{b}_i^\dagger(t) \hat{b}_j^\dagger \rangle \right| \left\| [\hat{b}_i(t), \hat{b}_j] \right\| \\ &\quad + \left| \langle \hat{b}_j \hat{b}_i(t) \rangle \right| \left\| [\hat{b}_i^\dagger(t), \hat{b}_j^\dagger] \right\| + \left| \langle \hat{b}_j^\dagger \hat{b}_i(t) \rangle \right| \left\| [\hat{b}_i^\dagger(t), \hat{b}_j] \right\|, \end{aligned} \quad (50)$$

where bounds on the commutators may then be obtained by identifying bosonic operators by canonical operators through Eq. (5) and employing the above derived bounds.

3 Proofs

In this section, we will present in detail the proofs of the previous statements.

3.1 Preliminaries

We write the Hamiltonian in Eq. (4) as

$$\hat{H} = \frac{1}{2} \sum_{n_i, n_j=1}^{2|L|} \hat{r}_{n_i} H_{n_i, n_j} \hat{r}_{n_j}, \quad (51)$$

where we have arranged lattice sites such that $H_{n_i, n_j} = X_{i,j}$ ($= P_{i,j}$) for $1 \leq n_i, n_j \leq |L|$ ($L+1 \leq n_i, n_j \leq 2|L|$) and $\hat{r}_{n_i} = \hat{x}_i$ ($= \hat{p}_i$) for $1 \leq n_i \leq |L|$ ($L+1 \leq n_i \leq 2|L|$). Now consider the time evolution of the operator

$$\hat{r}_{n_i}(t) := e^{i\hat{H}t} \hat{r}_{n_i} e^{-i\hat{H}t}. \quad (52)$$

By solving Heisenberg's equation of motion or, alternatively, by employing the Baker-Hausdorff formula, one finds

$$\hat{r}_{n_i}(t) = \sum_{n_j=1}^{2|L|} (e^{-\sigma H t})_{n_i, n_j} \hat{r}_{n_j}, \text{ where } \sigma_{n_i, n_j} = i [\hat{r}_{n_i}, \hat{r}_{n_j}]. \quad (53)$$

This yields for the commutator

$$i [\hat{r}_{n_i}(t), \hat{r}_{n_j}] = i \sum_{n_k=1}^{2|L|} (e^{-\sigma H t})_{n_i, n_k} [\hat{r}_{n_k}, \hat{r}_{n_j}] = (e^{-\sigma H t} \sigma)_{n_i, n_j}. \quad (54)$$

Now, separating the terms with an even power in n from the terms with an odd power, we get

$$\begin{aligned} e^{-\sigma H t} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{pmatrix} 0 & P \\ -X & 0 \end{pmatrix}^n \\ &= \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \begin{pmatrix} 0 & P \\ -X & 0 \end{pmatrix}^{2n+1} + \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \begin{pmatrix} 0 & P \\ -X & 0 \end{pmatrix}^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \begin{pmatrix} (PX)^n & 0 \\ 0 & (XP)^n \end{pmatrix} \begin{pmatrix} 0 & P \\ -X & 0 \end{pmatrix} \\ &\quad + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \begin{pmatrix} (PX)^n & 0 \\ 0 & (XP)^n \end{pmatrix}. \end{aligned} \quad (55)$$

Hence,

$$\begin{aligned}
i[\hat{x}_i(t), \hat{x}_j] &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} ((PX)^n P)_{i,j} =: \mathbb{1} \cdot C_{i,j}^{xx}(t), \\
i[\hat{p}_i(t), \hat{p}_j] &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} ((XP)^n X)_{i,j} =: \mathbb{1} \cdot C_{i,j}^{pp}(t), \\
i[\hat{x}_i(t), \hat{p}_j] &= - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} ((PX)^n)_{i,j} =: \mathbb{1} \cdot C_{i,j}^{xp}(t), \\
i[\hat{p}_i(t), \hat{x}_j] &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} ((XP)^n)_{i,j} =: \mathbb{1} \cdot C_{i,j}^{px}(t).
\end{aligned} \tag{56}$$

These expressions will form the starting point of the subsequent considerations.

3.2 Local couplings

We will need the following lemma. It states that finite powers of local coupling matrices defined on graphs remain local couplings, albeit with a larger range.

Lemma 1 (Products of local couplings) *Let $A = (A_{i,j})_{i,j \in L}$ be such that $A_{i,j} = 0$ for $\text{dist}(i, j) > R$. Then for $n \in \mathbb{N}$*

$$(A^n)_{i,j} = 0 \text{ for all } i, j \in L \text{ with } \text{dist}(i, j) > nR. \tag{57}$$

Proof. For $n = 1$ the statement is obviously true. Now let $(A^n)_{i,j} = 0$ for $\text{dist}(i, j) > nR$. Then

$$(A^{n+1})_{i,j} = \sum_{k \in L} (A^n)_{i,k} A_{k,j}. \tag{58}$$

Now let $k \in L$. If $\text{dist}(k, j) > R$ then this k does not contribute to the sum as $A_{k,j} = 0$. Now let $\text{dist}(i, j) > (n+1)R$. Then we have that also if $\text{dist}(k, j) \leq R$ it does not contribute to the sum as then $\text{dist}(i, k) > nR$ and therefore $(A^n)_{i,k} = 0$:

$$(n+1)R < \text{dist}(i, j) \leq \text{dist}(i, k) + \text{dist}(k, j) \leq \text{dist}(i, k) + R, \tag{59}$$

i.e., $\text{dist}(i, k) > nR$. \square

Thus, if we have $X_{i,j} = P_{i,j} = 0$ for $\text{dist}(i, j) > R$, we may write (see Eqs. (56)),

$$|C_{i,j}^{xx}(t)| \leq \sum_{n=a_{i,j}}^{\infty} \frac{|t|^{2n+1}}{(2n+1)!} |((PX)^n P)_{i,j}|, \tag{60}$$

where $a_{i,j} = \max\{0, \lceil (d_{i,j} - 1)/2 \rceil\}$ and we recall that $d_{i,j} = \text{dist}(i, j)/R$. As one has for any matrix that $|M_{i,j}| \leq \|M\|$, we find

$$|C_{i,j}^{xx}(t)| \leq \frac{\|P\|}{\sqrt{\|PX\|}} \sum_{n=a_{i,j}+1/2}^{\infty} \frac{\tau^{2n}}{(2n)!}, \tag{61}$$

where we recall that $\tau = \max\{\sqrt{\|PX\|}, \sqrt{\|XP\|}\}|t|$. Similarly ($b_{i,j} := \lceil d_{i,j}/2 \rceil$)

$$\begin{aligned} |C_{i,j}^{pp}(t)| &\leq \frac{\|X\|}{\sqrt{\|XP\|}} \sum_{n=a_{i,j}+1/2}^{\infty} \frac{\tau^{2n}}{(2n)!} \\ |C_{i,j}^{xp}(t)|, |C_{i,j}^{px}(t)| &\leq \sum_{n=b_{i,j}}^{\infty} \frac{\tau^{2n}}{(2n)!}. \end{aligned} \quad (62)$$

In the case of $P_{i,j} = \delta_{i,j}$ these bounds read

$$\begin{aligned} |C_{i,j}^{xx}(t)| &\leq \frac{1}{\sqrt{\|X\|}} \sum_{n=\lceil d_{i,j} \rceil + 1/2}^{\infty} \frac{\tau^{2n}}{(2n)!} \\ |C_{i,j}^{pp}(t)| &\leq \sqrt{\|X\|} \sum_{n=\max\{0, \lceil d_{i,j} \rceil - 1\} + 1/2}^{\infty} \frac{\tau^{2n}}{(2n)!} \\ |C_{i,j}^{xp}(t)|, |C_{i,j}^{px}(t)| &\leq \sum_{n=\lceil d_{i,j} \rceil}^{\infty} \frac{\tau^{2n}}{(2n)!}. \end{aligned} \quad (63)$$

Now, for $c \geq 0$,

$$\sum_{n=c}^{\infty} \frac{\tau^{2n}}{(2n)!} = \tau^{2c} \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n+2c)!} \leq \frac{\tau^{2c}}{(2c)!} \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n)!} = \frac{\tau^{2c}}{(2c)!} \cosh(\tau), \quad (64)$$

also, if $e\tau < 2c$, we find

$$\sum_{n=c}^{\infty} \frac{\tau^{2n}}{(2n)!} \leq \sum_{n=c}^{\infty} \left(\frac{e\tau}{2n}\right)^{2n} (2n)^{-1/2} \leq \frac{1}{\sqrt{2c}} \sum_{n=c}^{\infty} \left(\frac{e\tau}{2c}\right)^{2n} = \frac{(e\tau/2c)^{2c}}{\sqrt{2c}(1 - (e\tau/2c)^2)}, \quad (65)$$

where we have used that $n! \geq (n/e)^n n^{1/2}$ for $n \geq 1$.

3.3 Non-local couplings

Let $|M_{i,j}| \leq [1 + \text{dist}(i, j)]^{-\eta}$. For such couplings we have ($d_{i,j} := \text{dist}(i, j)$)

$$\begin{aligned} |(M^2)_{i,j}| &\leq (1 + d_{i,j})^{-\eta} \sum_k \left(\frac{1 + d_{i,j}}{(1 + d_{i,k})(1 + d_{k,j})} \right)^{\eta} \\ &\leq (1 + d_{i,j})^{-\eta} \sum_k \left(\frac{1 + d_{i,k} + 1 + d_{k,j}}{(1 + d_{i,k})(1 + d_{k,j})} \right)^{\eta} \\ &\leq \left(\frac{2}{1 + d_{i,j}} \right)^{\eta} \sum_k \left(\frac{1 + \max\{d_{i,k}, d_{k,j}\}}{(1 + d_{i,k})(1 + d_{k,j})} \right)^{\eta}, \end{aligned} \quad (66)$$

where we have used the triangle inequality and $(a + b) \leq 2 \max\{a, b\}$. Now,

$$(1 + \max\{d_{i,k}, d_{k,j}\})^{\eta} \leq (1 + d_{i,k})^{\eta} + (1 + d_{k,j})^{\eta}, \quad (67)$$

i.e., an upper bound for the above sum over k is given by

$$\begin{aligned} \sum_k \left(\frac{1}{(1+d_{i,k})^\eta} + \frac{1}{(1+d_{k,j})^\eta} \right) &\leq 2 \sup_{i \in L} \sum_k \frac{1}{(1+d_{i,k})^\eta} \\ &= 2 \sum_{r=0}^{\infty} \frac{\sup_{i \in L} \sum_k \delta_{r,d_{i,k}}}{(1+r)^\eta}, \end{aligned} \quad (68)$$

where

$$\sum_k \delta_{r,d_{i,k}} = |\{k \in L \mid d_{k,i} = r\}| = |S_r(i)|, \quad (69)$$

which we may bound using the definition of the dimension of the graph to find

$$|(M^2)_{i,j}| \leq c_D \frac{2^{\eta+1}}{(1+d_{i,j})^\eta} \sum_{r=0}^{\infty} \frac{1}{(1+r)^{\eta-D+1}}, \quad (70)$$

which converges if $\eta > D$, in which case we have

$$|(M^2)_{i,j}| \leq \frac{a_0}{(1+d_{i,j})^\eta}, \quad a_0 = c_D 2^{\eta+1} \zeta(1-D+\eta), \quad (71)$$

where ζ is the Riemann zeta function. By induction we then find

$$|(X^n)_{i,j}|, |(P^n)_{i,j}| \leq \frac{c_0^n a_0^{n-1}}{(1+d_{i,j})^\eta} \quad (72)$$

for $n \geq 1$, implying (recalling that $\tau = c_0 a_0 |t|$)

$$\begin{aligned} |C_{i,j}^{xx}(t)| &\leq \frac{1}{a_0(1+\text{dist}(i,j))^\eta} \sum_{n=0}^{\infty} \frac{\tau^{2n+1}}{(2n+1)!} = \frac{\sinh(\tau)}{a_0(1+\text{dist}(i,j))^\eta}, \\ |C_{i,j}^{xp}(t)| &\leq \delta_{i,j} + \frac{1}{a_0(1+\text{dist}(i,j))^\eta} \sum_{n=0}^{\infty} \frac{\tau^{2n}}{(2n)!} = \delta_{i,j} + \frac{\cosh(\tau)}{a_0(1+\text{dist}(i,j))^\eta}, \end{aligned} \quad (73)$$

and similarly

$$\begin{aligned} |C_{i,j}^{pp}(t)| &\leq \frac{\sinh(\tau)}{a_0(1+\text{dist}(i,j))^\eta}, \\ |C_{i,j}^{px}(t)| &\leq \delta_{i,j} + \frac{\cosh(\tau)}{a_0(1+\text{dist}(i,j))^\eta}. \end{aligned} \quad (74)$$

3.4 Weyl operators

For operators \hat{W}_ξ as above we find

$$\hat{W}_\xi(t) = e^{i \sum_{i \in \Xi} (p_i \hat{x}_i(t) - x_i \hat{p}_i(t))}, \quad \xi = (x_1, \dots, x_{|\Xi|}, p_1, \dots, p_{|\Xi|}) \in \mathbb{R}^{2|\Xi|} \quad (75)$$

Employing the Baker-Hausdorff identity we then have, see Eq. (56),

$$\hat{W}_\xi(t)\hat{W}_{\xi'} = \hat{W}_{\xi'}\hat{W}_\xi(t)e^{i\sum_{i\in\Xi,j\in\Xi'}(p_i p'_j C_{i,j}^{xx}(t) - p_i x'_j C_{i,j}^{xp}(t) - x_i p'_j C_{i,j}^{px}(t) + x_i x'_j C_{i,j}^{pp}(t))}, \quad (76)$$

i.e.,

$$\begin{aligned} \left\| \left[\hat{W}_\xi(t), \hat{W}_{\xi'} \right] \right\| &\leq \left\| e^{i\sum_{i\in\Xi,j\in\Xi'}(p_i p'_j C_{i,j}^{xx}(t) - p_i x'_j C_{i,j}^{xp}(t) - x_i p'_j C_{i,j}^{px}(t) + x_i x'_j C_{i,j}^{pp}(t))} - \mathbb{1} \right\| \\ &\leq \|\xi\| \|\xi'\| \sum_{i\in\Xi,j\in\Xi'} (|C_{i,j}^{xx}(t)| + |C_{i,j}^{xp}(t)| + |C_{i,j}^{px}(t)| + |C_{i,j}^{pp}(t)|) \\ &= \|\xi\| \|\xi'\| \sum_{i\in\Xi,j\in\Xi'} (\| [x_i(t), x_j] \| + \| [x_i(t), p_j] \| \\ &\quad + \| [p_i(t), x_j] \| + \| [p_i(t), p_j] \|) \\ &\leq \|\xi\| \|\xi'\| \sum_{i\in\Xi,j\in\Xi'} f(\text{dist}(i, j)), \end{aligned} \quad (77)$$

where

$$\sum_{i\in\Xi,j\in\Xi'} f(\text{dist}(i, j)) = \sum_{d=\text{dist}(\Xi,\Xi')}^{\infty} f(d) \sum_{i\in\Xi,j\in\Xi'} \delta_{\text{dist}(i,j),d}. \quad (78)$$

We now proceed by showing how to restrict the latter sum to subsets of Ξ and Ξ' . As one has to cross the boundary of a set to find a path to a site outside that set, there exist for all $i \in \Xi, j \in \Xi'$ sites $k \in \partial\Xi, l \in \partial\Xi'$ such that

$$\text{dist}(i, j) = \text{dist}(i, k) + \text{dist}(k, l) + \text{dist}(l, j). \quad (79)$$

Then $d = \text{dist}(i, j)$ requires $\text{dist}(i, k)$ and $\text{dist}(l, j)$ to be smaller than $d - \text{dist}(\Xi, \Xi') =: r$ as $\text{dist}(k, l) \geq \text{dist}(\Xi, \Xi')$. We may thus write

$$\sum_{i\in\Xi,j\in\Xi'} \delta_{\text{dist}(i,j),d} = \sum_{i\in\partial\Xi_r,j\in\partial\Xi'_r} \delta_{\text{dist}(i,j),d}, \quad (80)$$

where we denoted by

$$\partial A_r = \bigcup_{i\in\partial A} \{j \in A \mid \text{dist}(i, j) \leq r\} \quad (81)$$

the set of lattice sites that are within A and within a layer of thickness r around the surface of A , for which we have

$$\begin{aligned} |\partial A_r| &\leq |\partial A| \sup_{i\in\partial A} |\{j \in A \mid \text{dist}(i, j) \leq r\}| \\ &\leq |\partial A| \sup_{i\in L} |\{j \in L \mid \text{dist}(i, j) \leq r\}| \\ &= |\partial A| \sup_{i\in L} \sum_{l=0}^r |S_l(i)| \leq |\partial A| \left(1 + c_D \sum_{l=1}^r l^{D-1} \right). \end{aligned} \quad (82)$$

Hence

$$\begin{aligned}
\sum_{i \in \partial \Xi_r, j \in \partial \Xi'_r} \delta_{\text{dist}(i,j),d} &\leq \min \{ |\partial \Xi_r|, |\partial \Xi'_r| \} \sup_{j \in L} S_d(j) \\
&\leq c_D \min \{ |\partial \Xi_r|, |\partial \Xi'_r| \} d^{D-1} \\
&\leq c_D \min \{ |\partial \Xi|, |\partial \Xi'| \} d^{D-1} \left(1 + c_D \sum_{l=1}^r l^{D-1} \right) \\
&\leq c_D \min \{ |\partial \Xi|, |\partial \Xi'| \} d^{D-1} (1 + c_D r^D).
\end{aligned} \tag{83}$$

To summarize,

$$\begin{aligned}
\sum_{i \in \Xi, j \in \Xi'} f(\text{dist}(i,j)) &\leq c_D \min \{ |\partial \Xi|, |\partial \Xi'| \} \\
&\quad \times \sum_{d=\text{dist}(\Xi, \Xi')}^{\infty} f(d) d^{D-1} (1 + c_D (d - \text{dist}(\Xi, \Xi'))^D).
\end{aligned} \tag{84}$$

Under the assumptions of Theorem 2, e.g., we may choose

$$f(\text{dist}(i,j)) = \left(\frac{\|P\|}{\sqrt{\|PX\|}} + \frac{\|X\|}{\sqrt{\|XP\|}} + 2 \right) \frac{e^{d_{i,j} \log(e\tau/d_{i,j})}}{\sqrt{d_{i,j}} (1 - (e\tau/d_{i,j})^2)}, \tag{85}$$

i.e., for $e\tau < \text{dist}(\Xi, \Xi')/R =: d_{\Xi, \Xi'}$,

$$\begin{aligned}
\left\| [\hat{W}_\xi(t), \hat{W}_{\xi'}] \right\| &\leq \frac{c_D \|\xi\| \|\xi'\| \min \{ |\partial \Xi|, |\partial \Xi'| \} \left(\frac{\|P\|}{\sqrt{\|PX\|}} + \frac{\|X\|}{\sqrt{\|XP\|}} + 2 \right)}{\sqrt{d_{\Xi, \Xi'}} (1 - (e\tau/d_{\Xi, \Xi'})^2)} e^{d_{\Xi, \Xi'} \log(e\tau/d_{\Xi, \Xi'})} \\
&\quad \times \sum_{d=0}^{\infty} e^{d \log(e\tau R/(d + \text{dist}(\Xi, \Xi')))/R} (d + \text{dist}(\Xi, \Xi'))^{D-1} (1 + c_D d^D),
\end{aligned} \tag{86}$$

where we have for the sum the following upper bound ($z := e\tau/d_{\Xi, \Xi'}$)

$$\begin{aligned}
&\text{dist}^{D-1}(\Xi, \Xi') \sum_{d=0}^{\infty} e^{d \log(e\tau R/(d + \text{dist}(\Xi, \Xi')))/R} (d+1)^{D-1} (1 + c_D d^D) \\
&\leq \text{dist}^{D-1}(\Xi, \Xi') \sum_{d=0}^{\infty} z^{d/R} (d+1)^{D-1} (1 + c_D (d+1)^D).
\end{aligned} \tag{87}$$

4 Summary

In this work, we have presented Lieb-Robinson bounds for harmonic lattice systems on general lattices, complementing and generalizing work in Refs. [6] (see also Ref.

[33, 34]). We found a stronger than exponential decay in case of local interactions, and an inheritance of the decay behavior in case of algebraically decaying interactions. For the case of the Klein-Gordon field, we found the exact locality emerging from the approximate locality in the Lieb-Robinson sense. Specific attention was devoted to the time evolution of Weyl operators, which are an important class of operators in harmonic lattices. As such, this work provides a framework to study non-equilibrium dynamics in harmonic lattice systems in a general setting.

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References

- [1] E.H. Lieb and D.W. Robinson, Commun. Math. Phys. **28**, 251 (1972).
- [2] M.B. Hastings and T. Koma, Commun. Math. Phys. **265**, 781 (2006).
- [3] T. Koma, math-ph/0505022.
- [4] B. Nachtergaele, Y. Ogata, and R. Sims, J. Stat. Phys. **124** 1 (2006).
- [5] B. Nachtergaele and R. Sims, Commun. Math. Phys. **265**, 119 (2006).
- [6] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, arXiv:0712.3820.
- [7] P. Butta, E. Caglioti, S. Di Ruzza, and C. Marchioro, J. Stat. Phys. **127**, 313 (2007).
- [8] B. Nachtergaele and R. Sims, arXiv:0712.3318.
- [9] M.B. Hastings, JSTAT, P08024 (2007).
- [10] J. Eisert and T.J. Osborne, Phys. Rev. Lett. **97**, 150404 (2006).
- [11] S. Bravyi, M.B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
- [12] J. Eisert, M.B. Plenio, J. Hartley, and S. Bose, Phys. Rev. Lett. **93**, 190402 (2004).
- [13] M.B. Plenio, J. Hartley, and J. Eisert, New J. Phys. **6**, 36 (2004).
- [14] M.J. Hartmann, M.E. Reuter, and M.B. Plenio, New J. Phys. **8**, 94 (2006).
- [15] M. Cramer, C.M. Dawson, J. Eisert, and T.J. Osborne, Phys. Rev. Lett. **100**, 030602 (2008).
- [16] C. Kollath, A. Läuchli, and E. Altman, Phys. Rev. Lett. **98**, 180601 (2007).
- [17] K. Sengupta, S. Powell, and S. Sachdev, Phys. Rev. A **69**, 053616 (2004).
- [18] S.D. Huber, E. Altman, H.P. Büchler, and G. Blatter, Phys. Rev. B **75**, 085106 (2007).

- [19] V. Eisler and I. Peschel, J. Stat. Mech. P06005 (2007).
- [20] T. Barthel and U. Schollwöck, arXiv:0711.4896.
- [21] W.H. Zurek, U. Dorner, and P. Zoller, Phys. Rev. Lett. **95**, 105701 (2005).
- [22] G. De Chiara, S. Montangero, P. Calabrese, and R. Fazio, J. Stat. Mech. 0603, P001 (2006).
- [23] P. Calabrese and J. Cardy, Phys. Rev. Lett. **96**, 136801 (2006).
- [24] S. Fölling et al., Nature **448**, 1029 (2007).
- [25] M. Cramer and J. Eisert, New J. Phys. **8**, 71 (2006).
- [26] N. Schuch, J.I. Cirac, M.M. Wolf, Commun. Math. Phys. **267**, 65 (2006).
- [27] K. Audenaert, J. Eisert, M.B. Plenio, and R.F. Werner, Phys. Rev. A **66**, 042327 (2002).
- [28] M.B. Plenio, J. Eisert, J. Dreissig, and M. Cramer, Phys. Rev. Lett. **94**, 060503 (2005).
- [29] M. Cramer, J. Eisert, M.B. Plenio, and J. Dreissig, Phys. Rev. A **73**, 012309 (2006).
- [30] A. Botero and B. Reznik, Phys. Rev. A **70**, 052329 (2004).
- [31] M.M. Wolf, F. Verstraete, and J.I. Cirac, Phys. Rev. Lett. **92**, 087903 (2004).
- [32] M. Asoudeh and V. Karimipour, quant-ph/0506022.
- [33] O. Buerschaper, Diploma thesis (LMU Munich, Germany, 2007).
- [34] O. Buerschaper, M.M. Wolf, and J.I. Cirac, in preparation (2008).